A theoretical model of a voltage controlled oscillator

Yenming Chen and Robert A. Scholtz
Communication Sciences Institute
University of Southern California
Los Angeles, California 90089-2564, USA
E-mail: yenmingc@usc.edu, scholtz@usc.edu

***Abstract*— A novel mathematical model of a voltage controlled oscillator (VCO) based on physical dynamics with noise is proposed. The effects of noise on oscillators are shown and the analytical forms of the resulting phase noise are obtained using the stochastic integrals. It is shown that the VCO has phase noise contributed from the internal noise of the clock and the clock drift caused by tuning plus tracking loop noises. Moreover, a two-pole noise filter is designed to constrain tracking loop noise. The statistics of upward zero-crossing timing jitter are obtained as well as the cycle-to-cycle jitter. A timing jitter estimate is proposed. Analysis of the resulting phase noise along with the simulation suggest that the timing jitter process has a random walk behavior with restoring force and the upward zero-crossing jitter is normal distributed. The cycle-to-cycle jitter statistics are shown to be Gaussian distributed and independent.**

I. Introduction

In communication system analysis, where the voltage controlled oscillator (VCO) of the PLL is defined as a simple integrator, the model does not capture the essence of the dynamics of oscillation. Here, we propose a simple but novel mathematical model of a VCO based on physical dynamics with noise.

Extensive research has been done in the past for oscillators and standards. This includes the theoretical phase noise of oscillators based on structure functions [1]. Phase noise analysis is further investigated in electrical oscillators [2] and a positive feedback system approach [5] is used on a class of oscillators. Moreover, the approach to analyzing oscillator noise based on the stochastic integrals can be found in [4] where perturbation technique is used for noise analysis. In addition, in [3] an analysis and simulation of phase noise in VCO is investigated. However, most of the research are circuit-based and few address the effect of noise on a VCO.

We first introduce the proposed mathematical model of VCO with noise in Section II. An analytic solution of the differential equation without noise contribution with time varying input then follows in Section III. Analysis of noise effect on the proposed model with the presence of the internal noise of the clock is presented and the non-stationary phase noise process is derived in Section IV. When the tracking loop plus the controller noise is included for analysis on the model, a two-pole filter is introduced to constrain the tracking loop noise. We derive an analytic solution when both internal noise and tracking loop noise are present as well as the resulting phase noise description in Section V. It is then followed by simulation on the model and noise statistics analysis in Section VI. One of the main results is that the timing jitter process is shown to have a random walk behavior with restoring force. Further results such as cycle-to-cycle jitter and timing jitter estimate are given. Section VII draws the conclusions.

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II. Theoretical model

We propose a theoretical model of a voltage controlled oscillator based on physical dynamics with noise (Fig. 1). We show that a voltage controlled oscillator can be described by the following stochastic differential equation, which includes the internal noise, $F(t)$, of the clock and the tracking loop noise plus the controller noise, $n(t)$.

$$\ddot{y}-\frac{K_{\text{vco}}(\dot{v}+\dot{n}(t))}{\omega_0+K_{\text{vco}}(v+n(t))}\dot{y}+[\omega_0+K_{\text{vco}}(v+n(t))]^2y=c(t)F(t) \tag{1}$$

where we assume that $F(t)$ is a white Gaussian noise, which is independent of $n(t)$. In addition, $n(t)\in C^1([0,\infty))$, $y(t)$ is the oscillator generating waveform, $v(t)$ represents the controlled voltage, $c(t)$ is a scaled factor, K_{vco} is the VCO gain, and ω_0 is the clock rest frequency. The output of the proposed model after the hard-limiter is thus

$$Z(t)=\text{sgn}(y(t)). \tag{2}$$

III. Model without noise

First, we assume that $v(t)$ is known and the noise terms $n(t)$, $F(t)$ are not present. The initial conditions for $y(t)$, $v(t)$ are such that $y(0)=0$, $v(0)=0$. The stochastic differential equation then becomes an ordinary differential equation.

$$\ddot{y}-\frac{K_{\text{vco}}\dot{v}}{\omega_0+K_{\text{vco}}v}\dot{y}+[\omega_0+K_{\text{vco}}v]^2y=0 \tag{3}$$

Two cases are investigated. The first case is that $v(t)$ is a constant a when the steady state of the system is reached, i.e.,

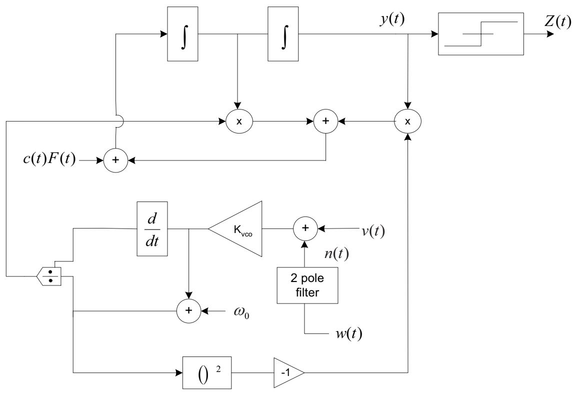

Fig. 1. Theoretical model of voltage controlled oscillator with thermal noise and noise from the controlled voltage and tracking loop.

when the error input voltage to the VCO is a constant. The equation (3) can then be solved to be

$$y(t) = A\sin(\omega_{\text{new}}t) \tag{4}$$

where $\omega_{\text{new}} = \omega_0 + K_{\text{vco}}a$, and A is any constant. For the case when $v(t)$ is any function other than a constant, equation (3) becomes a differential equation with time-varying coefficients. Suppose that $v(t)$ can be expressed as a ramp function

$$v(t) = \begin{cases} at & \text{if } t \le t_s \\ C & \text{if } t > t_s \end{cases} \tag{5}$$

where a, C are constants, and t_s is the time of steady state. Here no noise contribution is considered, equation (3) therefore becomes

$$\ddot{y} - \frac{K_{\text{vco}}a}{\omega_0 + K_{\text{vco}}at}\dot{y} + [\omega_0 + K_{\text{vco}}at]^2 y = 0 \tag{6}$$

for $t \le t_s$. Let $y_1 = y,\ y_2 = \dot{y}_1$, a state space equation can therefore be written as

$$\dot{\mathbf{Y}} = \mathbf{A}(t)\mathbf{Y} \tag{7}$$

where

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},\ \mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -(\omega_0 + K_{\text{vco}}at)^2 & \frac{K_{\text{vco}}a}{\omega_0 + K_{\text{vco}}at} \end{bmatrix}.$$

The above differential equation can be solved in various ways. By the existence and uniqueness theorems [6], the elements of matrix $\mathbf{A}(t)$ in (7) are continuous on an open interval $0 \le t < t_s$, containing the initial point $t = t_0$ (e.g., $t_0 = 0$), then there exists a unique solution, $y_1 = \phi_1(t),\ y_2 = \phi_2(t)$ of the system of differential equations (7). This set of solutions also satisfies the initial conditions, $y(0) = 0,\ v(0) = 0$. Furthermore, if the vector functions $y^{(1)},\ y^{(2)}$ are solutions of the system (7), then by the superposition principle, any linear combination $c_1 y^{(1)} +\ c_2 y^{(2)}$ is also a solution for any constants c_1 and c_2.

From the state space equation (7), the solution is found having the following form

$$\mathbf{Y} = \mathrm{c}_1\mathbf{y}^{(1)}(t) + \mathrm{c}_2\mathbf{y}^{(2)}(t). \tag{8}$$

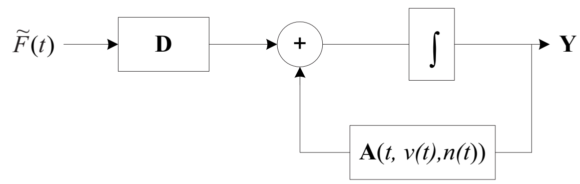

Fig. 2. Equivalent model of VCO (Fig. 1) with noise contributions, $\widetilde{F}(t)$, scaled internal noise, and $n(t)$, controller plus tracking loop noise.

The solution can be found such that $\mathbf{y}^{(1)}(t)$ and $\mathbf{y}^{(2)}(t)$ are linearly independent. They are

$$\mathbf{y}^{(1)}(t) = \begin{pmatrix} \sin(\omega_0 t + \int_0^t K_{\text{vco}}a\tau\, d\tau) \\ (\omega_0 + K_{\text{vco}}at)\cos(\omega_0 t + \int_0^t K_{\text{vco}}a\tau\, d\tau) \end{pmatrix},$$
$$\mathbf{y}^{(2)}(t) = \begin{pmatrix} -\cos(\omega_0 t + \int_0^t K_{\text{vco}}a\tau\, d\tau) \\ (\omega_0 + K_{\text{vco}}at)\sin(\omega_0 t + \int_0^t K_{\text{vco}}a\tau\, d\tau) \end{pmatrix}. \tag{9}$$

It can be shown that they satisfy equation (7). The Wronskian of $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ is greater than zero for $t \ge 0$. Therefore, the solution exists without discontinuity. For simplicity, when $c_1 = c_2 = 1$, the solution that is verified to be periodic in the limit is

$$y_1(t) = \sin(\omega_0 t + \int_0^t K_{\text{vco}}v(\tau)\, d\tau). \tag{10}$$

A numerical method is used to demonstrate the behavior of the solution for equation (7). It has been verified that the numerical solution agrees with the solution derived. As a result, for any other controlled voltage waveform with a steady state value in the limit, the solution becomes periodic in the limit.

IV. Model with Noise

In general, the proposed model shown in Fig. 1 can be represented by the equivalent model shown in Fig. 2. Two cases are considered in the following two sections. One occurs when the internal noise, $F(t)$, of the clock is present while $n(t)$ is excluded from the calculation. A stochastic differential equation is thus obtained,

$$\dot{\mathbf{Y}} = \mathbf{A}(t)\mathbf{Y} + \mathbf{D}\widetilde{F}(t)\ , \tag{11}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -(\omega_0 + K_{\text{vco}}v)^2 & \frac{K_{\text{vco}}\dot{v}}{\omega_0 + K_{\text{vco}}v} \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\widetilde{F}(t) = c(t)F(t)\ .$$

The equation (11) can be solved using $It\hat{o}\ integral$. We considered the case when the controlled voltage $v(t) = a$, where a is a constant. Similarly, it can be extended to the case of time-varying $v(t)$ as well. The solution to equation (11) with initial conditions that $y_1(0) = 0,\ y_2(0) = \hat{a}\omega_{\text{new}}$, where $\hat{a} \in \mathbf{R}$ is

$$y_1(t) = \hat{a}\sin(\omega_{\text{new}}t) + \int_0^t \frac{c(s)\sin(\omega_{\text{new}}(t-s))}{\omega_{\text{new}}}\, dB_s \tag{12}$$

where $c(s)$ is a scaling factor for the noise, $F(t)$, and B_s is a 1-dimensional Brownian motion.

1) noise analysis: Without loss of generality, let $\hat{a} = 1$, then equation (12) can then be expressed as

$$y_1(t) = \sin \omega_{\text{new}} t + n_1(t) \sin \omega_{\text{new}} t - n_2(t) \cos \omega_{\text{new}} t \quad (13)$$

where

$$n_1(t) = K \int_0^t \cos \omega_{\text{new}} s \, dB_s \ , \ n_2(t) = K \int_0^t \sin \omega_{\text{new}} s \, dB_s \ ,$$

and $K = \frac{c}{\omega_{\text{new}}}$ for $c(s) = c$. Using complex notation we can write for equation (13)

$$\begin{aligned} y_1(t) &= \text{Re}\{-j[(1 + n_1(t)) - jn_2(t)]\ e^{j(\omega_{\text{new}} t)}\} \\ &= \text{Re}\{-j\sqrt{(1 + n_1(t))^2 + n_2^2(t)}\ e^{j(\omega_{\text{new}} t + \varphi(t))}\} \\ &= \sqrt{(1 + n_1(t))^2 + n_2^2(t)} \sin(\omega_{\text{new}} t + \varphi(t)) \quad (14) \end{aligned}$$

with

$$\varphi(t) = \tan^{-1}\left(\frac{n_2(t)}{1 + n_1(t)}\right) \ , \quad (15)$$

where $\varphi(t)$ is the oscillator phase noise contributed by the scaled internal noise, $\widetilde{F}(t)$, of the oscillator.

2) Noise processes: The noise processes, $n_1(t)$, $n_2(t)$, have the following statistical properties. Both noise processes are zero mean, Gaussian distributed, and non-stationary. The correlation functions for $n_1(t)$ and $n_2(t)$ are found to be

$$\begin{aligned} R_{n_1}(t, s) &= E[n_1(t)n_1(s)] \\ &= \frac{c^2}{2\omega_{\text{new}}^2}\left[\min(s, t) + \frac{\sin(2\omega_{\text{new}} \min(s, t))}{2\omega_{\text{new}}}\right], \\ R_{n_2}(t, s) &= E[n_2(t)n_2(s)] \\ &= \frac{c^2}{2\omega_{\text{new}}^2}\left[\min(s, t) - \frac{\sin(2\omega_{\text{new}} \min(s, t))}{2\omega_{\text{new}}}\right], \end{aligned}$$

and the cross-correlation function is found to be

$$\begin{aligned} R_{n_1,n_2}(t, s) &= E[n_1(t)n_2(s)] \\ &= \frac{c^2}{4\omega_{\text{new}}^3}[1 - \cos(2\omega_{\text{new}} \min(s, t))] \ . \end{aligned}$$

In addition, $\dot{n}_1(t)$ is found to be wide-sense cyclo-stationary, similarly, so is $\dot{n}_2(t)$. When the 2 $\times$ frequency terms in the correlation functions are filtered, $\dot{n}_1(t)$, $\dot{n}_2(t)$ are strict sense stationary. Furthermore, the first increment process of $n_1(t)$, $n_2(t)$, are found to be non-stationary.

If we denote the noise effect $\widetilde{F}(t)$ on the VCO by

$$N_{\widetilde{F}}(t) = \int_0^t c(s) \frac{\sin(\omega_{\text{new}}(t - s))}{\omega_{\text{new}}} dB_s \ , \quad (16)$$

the variance of $N_{\widetilde{F}}(t)$ is found to be

$$\sigma_{N_{\widetilde{F}}}^2 = \frac{\sigma^2 c^2}{4\omega_{\text{new}}^2}[2t - \frac{\sin(2\omega_{\text{new}} t)}{\omega_{\text{new}}}] \quad (17)$$

where σ^2 is the noise power of $F(t)$. This noise generates the amplitude and phase noise of a VCO before the hard-limiter.

3) phase noise process: The phase noise process (15) induced by the scaled internal noise, $\widetilde{F}(t)$, can be approximated by the following equation

$$\varphi(t) \approx \frac{n_2(t)}{1 + n_1(t)} \approx n_2(t)(1 - n_1(t)) \ . \quad (18)$$

It has been shown that this is a valid approximation by simulation for low noise level. Furthermore, the phase noise $\varphi(t)$ has the following statistical properties. The mean is given by

$$\begin{aligned} E[\varphi(t)] &\approx E[n_2(t) - n_2(t)n_1(t)] \\ &= \frac{c^2}{4\omega_{\text{new}}^3}[\cos(2\omega_{\text{new}} t) - 1] \quad (19) \end{aligned}$$

with the correlation function

$$R_{\varphi}(t, s) = E[\varphi(t)\varphi(s)] \approx R_{n_2}(t, s) \quad (20)$$

when the time step $\Delta t \rightarrow 0$ from the derivation by a stochastic integral.

The phase noise process $\varphi(t)$ can also be approximated by $n_2(t)$ for low noise level and it is shown as a valid assumption by simulation. In this case, the phase noise process $\varphi(t)$ has the same statistical property as $n_2(t)$, meaning it is Gaussian distributed and $\dot{\varphi}(t)$ is a wide-sense stationary process.

V. MODEL WITH $n(t)$ PRESENT

The original proposed VCO model (1) has an additional noise contribution, $n(t)$, from the tracking loop plus the controller. In addition, $n(t) \in C^2([0, \infty))$, and a 2 pole filter is proposed to constrain the tracking loop noise due to the differentiator as shown in Fig. 1.

We considered a two pole filter when $n(t)$ is present (shown in Fig. 3), where $W(t) = W_{\text{contr}}(t) + W_{\text{loop}}(t)$ is white assuming noise processes $W_{\text{contr}}(t)$ and $W_{\text{loop}}(t)$ are independent.

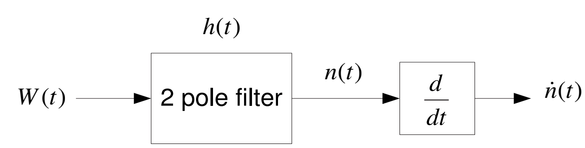

Fig. 3. General scheme for the tracking loop plus the controller noise, $n(t)$

The transfer function of a general 2-pole filter is

$$H(s)|_{s=j\omega} = \frac{\omega_{\text{n}}^2}{s^2 + 2\zeta\omega_{\text{n}} s + \omega_{\text{n}}^2} \ ,$$

where ζ is a damping factor, and ω_{n} filter natural frequency. Assuming that the averaged control output voltage after the filter $v(t)$ is equal to a constant, a, the voltage controller is therefore corrupted by the filtered noise, $n(t)$.

The filtered noise $n(t)$ can be expressed as the following state equations. Let $x_1(t) = \frac{n(t)}{\omega_{\text{n}}^2}$ and $x_2(t) = \dot{x}_1(t)$ under the condition that $n(t) \in C^2([0, \infty))$, the state equation is shown as

$$\dot{\mathbf{X}} = \widetilde{\mathbf{A}}\mathbf{X} + \widetilde{\mathbf{D}}\, W(t) \quad (21)$$

where

$$\widetilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \; \widetilde{\mathbf{D}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Analytical solution to the above equation (21) can be solved using a stochastic integral. It has the following form

$$\mathbf{X}(t) = \exp(\widetilde{\mathbf{A}}t)\mathbf{X}(0) + \int_0^t \exp(\widetilde{\mathbf{A}}(t-s))\widetilde{\mathbf{D}}\, dB_s \; , \quad (22)$$

where dB_s is the increment of a one-dimensional Brownian motion B_t. Assuming zero initial conditions for $\mathbf{X}(t)$, the correlation function of the output noise $n(t)$ can be found to be

$$R_n(t,\tau) = E[n(t)n(t+\tau)] = \int_0^t \sigma^2 h(l)h(l+\tau)\, dl \; , \quad (23)$$

where $h(t)$ is the impulse response of the filter, σ^2 is the input noise power of $W(t)$. When $t \to \infty$, the process $n(t)$ becomes stationary.

The analytical solution to the VCO model (11) when $\widetilde{F}(t)$, $n(t)$ are present and $v(t) = at \;\forall\; t$ with initial conditions of $y_1(0) = 0$, $y_2(0) = \hat{a}\omega_0$, where $\hat{a} \in \mathbf{R}$ is found using the fundamental matrix solution to be

$$y_1(t) = \hat{a}\sin\Phi_{\text{new}}(t) + \int_0^t \frac{c(s)\sin\hat{b}(s,t)}{\omega_0 + K_{\text{vco}}(as+n(s))} dB_s \quad (24)$$

where

$$\begin{aligned} \Phi_{\text{new}}(t) &= \omega_0 t + \int_0^t K_{\text{vco}} a\tau d\tau + \int_0^t K_{\text{vco}} n(\tau) d\tau \; , \\ \hat{b}(s,t) &= \omega_0(t-s) + \int_s^t K_{\text{vco}} a\tau d\tau + \int_s^t K_{\text{vco}} n(\tau) d\tau. \end{aligned}$$

1) noise analysis: Without loss of generality, let $\hat{a} = 1$, we can rewrite equation (24) as

$$y_1(t) = \sin\Phi_{\text{new}}(t) + \tilde{n}_1(t)\sin\Phi_{\text{new}}(t) - \tilde{n}_2(t)\cos\Phi_{\text{new}}(t) \quad (25)$$

where

$$\begin{aligned} \tilde{n}_1(t) &= \int_0^t \tilde{K}(s,n(s))\cos\Phi_{\text{new}}(s) dB_s \; , \\ \tilde{n}_2(t) &= \int_0^t \tilde{K}(s,n(s))\sin\Phi_{\text{new}}(s) dB_s \; , \end{aligned} \quad (26)$$

and $\tilde{K}(s,n(s)) = \frac{c}{\omega_0 + K_{\text{vco}}(as+n(s))}$ for $c(s) = c$. Again, using complex notation we can write for equation (25) as

$$y_1(t) = \sqrt{[1+\tilde{n}_1(t)]^2 + \tilde{n}_2^2(t)}\,\sin(\Phi_{\text{new}}(t) + \tilde{\varphi}(t)) \quad (27)$$

where

$$\tilde{\varphi}(t) = \tan^{-1}\left(\frac{\tilde{n}_2(t)}{1+\tilde{n}_1(t)}\right) . \quad (28)$$

We have after the hard limiter as shown in Fig. 1,

$$z(t) = \text{sgn}(y_1(t)) = \text{sgn}[\sin(\Phi_{\text{new}}(t) + \tilde{\varphi}(t))]$$

where $\Phi_{\text{new}}(t) + \tilde{\varphi}(t)$ is the total oscillator phase with phase noises caused by $n(t)$ and $\widetilde{F}(t)$.

2) noise processes: The noise processes, $\tilde{n}_1(t)$, $\tilde{n}_2(t)$, have the following statistical properties. Both noise processes are zero mean and non-stationary. The correlation functions for $\tilde{n}_1(t)$ and $\tilde{n}_2(t)$ are found to be

$$\begin{aligned} R_{\tilde{n}_1}(t,s) &= E[\tilde{n}_1(t)\tilde{n}_1(s)] \\ &= \frac{1}{2}\left\{\int_0^{\min(s,t)} E[\tilde{K}^2(z,n(z)) + \cos(2\Phi_{\text{new}}(z))]dz\right\}, \\ R_{\tilde{n}_2}(t,s) &= E[\tilde{n}_2(t)\tilde{n}_2(s)] \\ &= \frac{1}{2}\left\{\int_0^{\min(s,t)} E[\tilde{K}^2(z,n(z)) - \cos(2\Phi_{\text{new}}(z))]dz\right\}, \end{aligned}$$

and the cross-correlation function is found to be

$$\begin{aligned} R_{\tilde{n}_1,\tilde{n}_2}(t,s) &= E[\tilde{n}_1(t)\tilde{n}_2(s)] \\ &= \frac{1}{2}\int_0^{\min(s,t)} E[\tilde{K}^2(z,n(z))\sin(2\Phi_{\text{new}}(z))]dz \; . \end{aligned}$$

3) phase noise processes: For the tracking loop plus controller noise $n(t)$, the phase noise generated is denoted by

$$\varphi_n(t) = \int_0^t K_{\text{vco}} n(z) dz.$$

For phase noise generated by internal noise, $\widetilde{F}(t)$, of the VCO, $\tilde{\varphi}(t)$ from equation (28) can be approximated by the following equation for small noise level

$$\tilde{\varphi}(t) \approx \frac{\tilde{n}_2(t)}{1+\tilde{n}_1(t)} \approx \tilde{n}_2(t)(1-\tilde{n}_1(t)) \; . \quad (29)$$

Furthermore, the phase noise $\tilde{\varphi}(t)$ has the following statistical properties. The mean is given by

$$\begin{aligned} E[\tilde{\varphi}(t)] &\approx E[\tilde{n}_2(t) - \tilde{n}_2(t)n_1(t)] \\ &= -\frac{1}{2}\int_0^{min(s,t)} E[\tilde{K}^2(z,n(z))\sin(2\Phi_{\text{new}}(z))]dz \end{aligned} \quad (30)$$

with the correlation function

$$R_{\tilde{\varphi}}(t,s) = E[\tilde{\varphi}(t)\tilde{\varphi}(s)] \approx R_{\tilde{n}_2}(t,s) \quad (31)$$

when the time step $\Delta t \;\to 0$ from the derivation by a stochastic integral.

VI. Simulation Results and Timing Jitter Estimate

We performed a simulation on the proposed mathematical VCO model and we verified the analytical solutions and noise statistics as well by simulation. The simulation is done for two cases, 1) when the internal noise of the clock, $\widetilde{F}(t)$, is present, 2) when the tracking plus the controller noise, $n(t)$, is present as well. It is simulated at a VCO frequency of 2 kHz with noise power $\sigma_{\widetilde{F}}^2 = 1$, the scaling factor $c(t) = 1$, $v(t) = 1$, and $K_{\text{vco}} = 200\pi$ rad/V for 100 seconds. We obtained the averaged upward zero-crossing jitter statistics shown in Fig. 4, which indicates that the timing jitter has a random walk behavior with restoring force. Statistics from simulations also show that the timing jitter has a normal distribution at different times with a diffusion-like variance.

In addition to the timing jitter, we are interested in the cycle-to-cycle jitter statistics as shown in Fig. 5. We performed

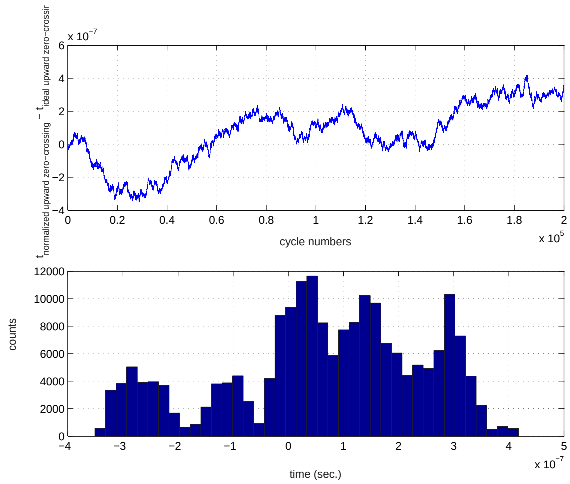

Fig. 4. Averaged upper zero-crossing jitter for a VCO with 2 kHz clock frequency with noise power of $\sigma^2_{\widetilde{F}} = 1$

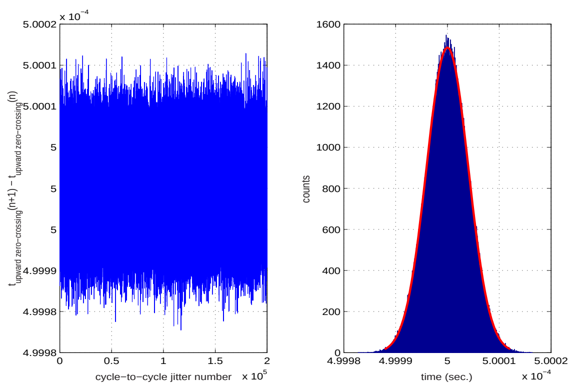

Fig. 5. Cycle-to-cycle jitter statistics for 100 seconds

the chi-squared goodness of fit test as a normality test for the test statistics. We obtained a p value of 0.73537 at 95% confidence interval such that we accept the hypothesis that it is normal distributed. We then performed the parametric Pearson correlation test on the test statistics. The results indicate that for p value < 0.05 at 95% confidence interval, we found that samples of test statistics are uncorrelated. We then conclude that they are independent since they are normal and uncorrelated.

When both noise sources $\widetilde{F}(t)$ and $n(t)$ are considered in the simulation, we see that at some time in the future, the timing jitter will drift away as shown in Fig. 6. It is caused by the tracking loop plus controller noise $n(t)$. The simulation was done for 100 seconds at $\sigma^2_W = 0.1$ with the Butterworth filter having $\zeta = 1.25$ and $\omega_{\text{n}} = 2$.

We then performed a timing jitter estimate based on the Markov property of the phase noise. We denote the total phase as $\Phi_{\text{tot}}(t) = \Phi_{\text{new}}(t) + \tilde{\varphi}(t) = f(\varphi_n(t), \tilde{\varphi}(t))$, where $f(\cdot)$ is any function. It is a Markov process since $\varphi_n(t)$ and $\tilde{\varphi}(t)$ are hidden Markov process. We have $y_1(t) = g(\Phi_{\text{tot}}(t)) = \sin(\Phi_{\text{tot}}(t))$ as a Markov process as well, where $g(\cdot)$ is any

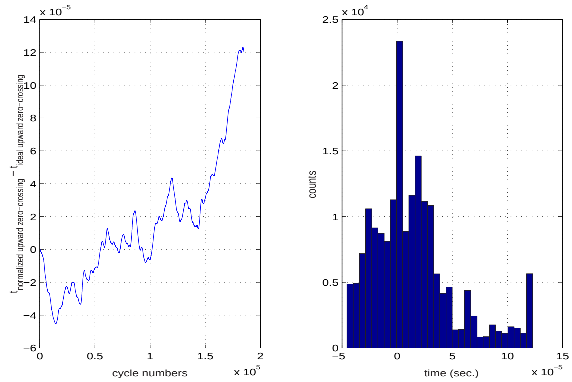

Fig. 6. Timing jitters when $\widetilde{F}(t)$ and $n(t)$ are present

function. We then take samples of $\Phi_{\text{tot}}(t)$ as $\Phi_{\text{tot}}(t_{\text{rise}}(k)) = 2\pi k = h_k$, where $k \in Z^+$. We obtain the normalized upward zero-crossing time estimate as $\Delta\hat{t}(k) = E[\Delta t(k)|\Phi_{\text{tot}}(t)|_0^T]$ where $\Delta t(k) = t_{\text{rise}}(k) - \frac{k}{f_{\text{new}}}$ and T is the stopping time.

VII. Conclusion

This paper describes a novel mathematical model of a voltage controlled oscillator based on physical dynamic with noise. The effects of noise on the proposed model are analyzed and the resulting phase noise is investigated. Analytical forms of the VCO with noise are obtained using a stochastic integral. Moreover, a two-pole filter is introduced for the constraint of tracking loop noise. Simulation of the model is done to further verify the analytic solutions and noise statistics. Analysis of the resulting phase noise along with the simulation suggest that the timing jitter process has a random walk behavior with restoring force and the upward zero crossing jitter is normal distributed. The increment of the timing jitter process, cycle-to-cycle jitter, is shown to behave as normal distributed and independent. We can conclude that from the results obtained, that the tracking loop plus the controller noise $n(t)$ cause the oscillator phase to drift while the internal noise $\widetilde{F}(t)$ tends to cause diffusion on the oscillator phase.

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